JOURNAL OF ECONOMIC THEORY 40, 229–249 (1986)

On the Faustmann Solution to the Forest Management Problem*

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Received August 5, 1982; revised November 15, 1985

This paper is concerned with optimal solutions to the forest management problem when future utilities are undiscounted. By examining asymptotic properties of such solutions, we find that (i) if the utility function is linear, then the Faustmann periodic solution is optimal; (ii) if the utility function is increasing and strictly concave, an optimal solution converges to the maximum sustained yield solution, which we characterize as a golden rule. These results may be viewed as a possible resolution to the debate in forestry economics about what constitutes an optimal policy in forest management. *Journal of Economic Literature* Classification Numbers: 111, 721.

1. INTRODUCTION

Consider an economy with an empty tract of land, which can be used for growing trees of a particular type. The age of a tree (a) determines the timber content of the tree ($f(a)$), through a given function, f. The utility of the economy in any time period is determined by the timber content of trees harvested in that period. If the economy has a discount rate of $\rho \geqslant 0$, and wishes to "maximize" the undiscounted or discounted sum of utilities, what pattern of planting and harvesting trees should it follow?

Assuming that one were interested in maximizing the discounted or undiscounted sum of the timber content of trees harvested (or, what is the same thing, assuming that the utility function is linear), Faustmann [4] suggested the following "periodic solution" to the problem posed above. The whole land should be planted with seedlings initially, and all seedlings should be allowed to grow to a certain age (M), at which time the whole forest should be cut down and replanted with seedlings. This process

* We are grateful to Murray Kemp for introducing us to the problems in forestry economics. Research of the first author was supported by a National Science Foundation Grant, an Alfred P. Sloan Research Fellowship, and a grant from Resources for the Future. The present version has benefited considerably from the comments of a referee of this journal.

0022-0531/86 $3.00

should be repeated indefinitely. Furthermore, the trees should be cut at the age (M) at which the increase in the timber content of the standing trees over an additional unit time period equals the sum of the following two factors: (i) the interest that can be earned if the timber content from cutting the trees is invested at an interest rate of ρ; (ii) the interest that can be earned on the "site value" (that is, on the present value of the stream of all future harvests on the particular site) of the land released by cutting the trees. Contributions to capital theory by economists like Jevons [8] and Wicksell [20] suggests a solution to the tree-cutting problem which ignored the aspect of the "site-value."

Even though Faustmann suggested a periodic solution to the problem of optimal exploitation of forests, "there has been a tradition in forestry management which claims that the goal of good policy is to have sustained forest yield, or even maximum sustained yield somehow defined" (Samuelson [16, p. 146]). (The Faustmann model has been discussed extensively by economists like Gaffney [5], Pearse [13], and Scott [18]. More through studies on the economics of forestry are contained in Schreuder [17], Gregory [18], and Wan [19]. A very readable updated account can be found in Dasgupta [3].)

In this paper we consider optimal solutions to the problem posed in the first paragraph, when future utilities are undiscounted. By examining the asymptotic properties of such solutions, we find that (i) if the utility function is linear, then the Faustmann periodic solution is indeed optimal; (ii) if the utility function is increasing and strictly concave, an optimal solution converges to the "maximum sustained yield" solution, which we characterize as a "golden rule." Thus our paper provides a resolution of the debate in forestry economics about what constitutes an optimal policy in forest management. (It should be pointed out that we are not analyzing, in this paper, the case in which future utilities are positively discounted ($\rho > 0$). For a brief discussion of this case, see Sect. 7.)

The plan of the paper is the following: After setting up the model in Section 2, we look at "stationary forests" in Section 3, and characterize a golden rule as a stationary forest with maximum per period utility (equivalently, maximum per period yield). This corresponds to the notion of "maximum sustained yield" used in the forestry literature. We prove the existence of a stationary shadow price, which "supports" the golden-rule program in the sense that at this price, the sum of utility plus the value of timber stands carried over, less the value of initial timber stands is maximized at the golden-rule activity among all feasible activities. Our analysis shows that the golden-rule forest is one in which the total plot of land is split up into M equal sub-plots, with one sub-plot each containing trees of age a ($a = 0, \ldots, M-1$). In each period, trees of age M are cut down, and the sub-plot so cleared is planted with seedlings (age zero trees). It is of interest to note that the age at which trees are cut at the golden rule (M) is the same as the age at which trees are cut on the above-mentioned periodic Faustmann solution.

Since future utilities are undiscounted in our framework, we provide in Section 4 a proof of the existence of an optimal program, following the approaches of Gale [6], McKenzie [11], and Brock [1] in the theory of optimal intertemporal resource allocation.

In Section 5, we consider the case of a linear utility function, and show (i) if the plot of land is initially empty, then the above-mentioned Faustmann periodic solution is optimal; (ii) if the plot of land has initially a standing forest then the following rule is optimal: initially, cut all trees of age M or more; thereafter, cut a tree if and only if it is of age M. Note that this means that if we think of the land as divided into sub-plots, according to the age of trees standing on them, then each sub-plot follows the periodic Faustmann solution.

In Section 6, we consider the case of a strictly concave utility function, and show that the forest along an optimal program from "any initial forest" asymptotically approaches the golden-rule stationary forest. This result can be shown by following the general technique developed by McKenzie [11] in analyzing asymptotic convergence of optimal paths, using the concept of the "von Neumann Facet." We provide a direct proof for our special case in order to keep the exposition self-contained. The results of Sections 5 and 6 are related to the existing literature on optimal intertemporal allocation, particularly to the results and methods of Brock [1], Gale [6], and McKenzie [11].

2. The Model

2a. *Production*

Consider a framework in which the timber content of a tree is related to the age of the tree, through a production function, f, from R_+ to R. Given the age of a tree (a), the timber content of the tree is given by $f(a)$, for $a \geqslant 0$.

The following assumptions on f are used in the paper:

(A.1) *$f(a) = 0$ for $0 \leqslant a \leqslant \mathbf{a}$, for some $\mathbf{a} \geqslant 1$.*

(A.2) *f is continuous for $a \geqslant \mathbf{a}$, and there is a positive integer $N > \mathbf{a}$, such that* (i) *f is increasing for $\mathbf{a} \leqslant a < N$;* (ii) *$f$ is decreasing for $a > N$.*

2b. *Some Notation*

In specifying our notation, N will refer to the positive integer of Section 2a. Let d denote the first unit vector, and e the $(N+1)$th unit vector of

R^{N+1}; i.e., $d=(1, 0,..., 0)$, $e=(0, 0,..., 1)$ in R^{N+1}. Let μ be the sum vector in R^N (i.e., $\mu=(1, 1,..., 1)$ in R^N); v be the sum vector in R^{N+1}. Let I_N denote the $N\times N$ identity matrix. Define a $(N+1)\times(N+1)$ matrix

$$A=\begin{bmatrix} 0 & 1 \\ I_N & 0 \end{bmatrix}.$$

Define a $N\times(N+1)$ matrix B by

$$B=[0 \quad I_N].$$

Define a set D as follows: $D=[x \text{ in } R_+^{N+1};\ vx=1,\ ex=0]$. Define a set E as follows: $E=[(x, y) \text{ in } D\times R_+^{N+1};\ y=Ax]$. Note that for (x, y) in E, $vy=1$, and $dy=0$. Finally, define a set F as follows: $F=[(x, z) \text{ in } D\times D$: $B(Ax-z)\geqslant 0]$. Note that if (x, z) is in F, then $\mu B(Ax-z)=dz$.

2c. *Programs*

A *feasible program* from $\mathbf{x}$ in D, is a sequence $\langle x_t, y_{t+1}\rangle$ satisfying

$$x_0=\mathbf{x}, \quad (x_t, y_{t+1})\in E, \quad B(y_{t+1}-x_{t+1})\geqslant 0 \qquad \text{for} \quad t\geqslant 0. \tag{2.1}$$

Associated with a feasible program $\langle x_t, y_{t+1}\rangle$ from $\mathbf{x}$ in D, is a sequence $\langle c_{t+1}\rangle$ such that

$$c_{t+1}=B(y_{t+1}-x_{t+1}) \qquad \text{for} \quad t\geqslant 0. \tag{2.2}$$

By the properties of sets E and F noted in Section 2b, we have

$$vy_{t+1}=1, \qquad dy_{t+1}=0, \qquad \mu c_{t+1}=dx_{t+1} \qquad \text{for} \quad t\geqslant 0. \tag{2.3}$$

A feasible program $\langle x_t, y_{t+1}\rangle$ is *stationary* if $x_t=x_{t+1}$ for $t\geqslant 0$. In this case, we denote the stationary levels of x_t and y_{t+1} respectively by x and y, and the stationary value of c_{t+1} by c; that is, $c=B(y-x)=B(Ax-x)$. The feasible program itself is then denoted by $\langle x, y\rangle$.

We now provide some interpretation of the above definition for a feasible program. For a feasible program $\langle x_t, y_{t+1}\rangle$, let $x_t=[x_t(0),..., x_t(N)]$; then $x_t(a)$, for $a=0, 1,..., N$, is the land occupied by *input* of trees of age a, at the end of time period t. The total amount of land available for forestry in the economy is assumed to be one unit, so $vx_t=1$. Also, for any reasonable objective function for the economy, trees will never be allowed to grow beyond age N; we therefore take this as a condition of feasibility itself. That is, without loss of generality, feasible programs can be restricted to those satisfying $x_t(N)=0$, or equivalently, $ex_t=0$. Thus, x_t belongs to the set D for each t.

Let $y_{t+1}=[y_{t+1}(0),..., y_{t+1}(N)]$; then $y_{t+1}(a)$, for $a=0, 1,..., N$, is the land occupied by *output* of trees of age a, at the end of time period $(t+1)$.

Since in one period a tree of age (a) becomes a tree of age $(a+1)$, so $y_{t+1}(1)=x_t(0);...;y_{t+1}(N)=x_t(N-1)$. Furthermore, $y_{t+1}(0)$ is, by definition, equal to zero, that is, $dy_{t+1}=0$. Thus, we have $y_{t+1}=Ax_t$, and (x_t, y_{t+1}) is in the set E. Note that as a consequence we have $vy_{t+1}=1$, which simply reflects the fact tht the total amount of land available for forestry is one unit.

At the end of time period $(t+1)$, two things are supposed to happen instantaneously, by the nature of our "point-input, point-output" framework. First, trees of different ages are cut down. Second, new seedlings (trees of age zero) are planted in the cleared areas. Let $x_{t+1}=[x_{t+1}(0),..., x_{t+1}(N)]$; then $x_{t+1}(a)$, for $a=0, 1,..., N$, is the land occupied by input of trees of age a, at the end of time period $(t+1)$. Then, clearly, $y_{t+1}(1)\geqslant x_{t+1}(1),..., y_{t+1}(N)\geqslant x_{t+1}(N)$. This means that $B(y_{t+1}-x_{t+1})\geqslant 0$.

Let $c_{t+1}=[c_{t+1}(1),..., c_{t+1}(N)]$; then $c_{t+1}(a)$, for $a=1,..., N$, is the land released by *harvest* of trees of age a, at the end of time period $(t+1)$. Note then that $c_{t+1}(a)$ is precisely measured by $(y_{t+1}(a)-x_{t+1}(a))$ for $a=1,..., N$. Thus we have $c_{t+1}=B(y_{t+1}-x_{t+1})$. Since input of trees of age zero at the end of time period $(t+1)$ occupy the land released by all harvests, $x_{t+1}(0)=c_{t+1}(1)+\cdots+c_{t+1}(N)$; that is, $\mu c_{t+1}=dx_{t+1}$.

We have now explained (2.1), (2.2), and (2.3) as the economy moves from the end of time period t to the end of time period $(t+1)$. The above process is then repeated indefinitely.

2d. *Preferences*

Preferences are represented by a utility function, u, from R_+ to R. The following assumptions on u are used in the paper:

(A.3) *u is strictly increasing.*

(A.4) *u is continuous on R_+ and differentiable on R_{++}.*

(A.5) *u is concave.*

Define $Q=[f(1),..., f(N)]$. A feasible program $\langle x_t^*, y_{t+1}^*\rangle$ from **x** in D is called *optimal* if

$$\liminf_{T\to\infty} \sum_{t=1}^{T} [u(Qc_t)-u(Qc_t^*)]\leqslant 0 \tag{2.4}$$

for every feasible program $\langle x_t, y_{t+1}\rangle$ for **x** in D. (Here, we are adopting a somewhat different terminology than that used in the traditional theory of optimal intertemporal allocation. What we call "optimal" is labeled as "weakly maximal" by Brock [1]. Gale [6] calls a program $\langle x_t^*, y_{t+1}^*\rangle$ "optimal" if (2.4) holds with "lim inf" replaced by "lim sup.")

We turn now to an interpretation of this definition. For a feasible program $\langle x_t, y_{t+1} \rangle$ from $\mathbf{x}$ in D, $c_t(a)$ for $a = 1,..., N$ is the land released by harvest of trees of age a, at the end of time period t. Assuming that the trees on a plot of land are proportional to the amount of land (the factor of proportionality being unity by suitable choice of units in which the number of trees are measured), the timber content obtained by harvest at the end of time period t is given by $[f(1)\, c_t(1) + \cdots + f(N)\, c_t(N)]$, or, equivalently, by Qc_t. The function u, then, measures the utility obtained from this timber content at the end of time period t, $u(Qc_t)$. Implicitly, costs of planting and harvesting trees are being assumed to be zero, so that these costs do not enter as arguments in the utility function. If the utilities obtained in successive periods are not discounted (in the interests of integenerational equity, following Ramsey [15]) and the sum of such utilities is to be "maximized," in the sense of the well-known overtaking criterion, one can define an optimal program $\langle x_t^*, y_{t+1}^* \rangle$ to be one which cannot be "overtaken" by a fixed positive amount by any other feasible program from the same initial condition; that is, by (2.4).

The theory of optimal forest management as we will present it here will be an application of the results and methods of the general theory of optimal intertemporal allocation as developed by Gale [6], McKenzie [11], and Brock [1]. To conveniently relate the forestry theory to intertemporal allocation theory, the following notation will be useful.

Define a *welfare function*, $w: F \to R$, by $w(x, z) = u(QB(Ax - z))$ for (x, z) in F. (This is the utility achieved if this period's input is x and the next period's input is z, where z is "technologically" feasible from x in one period.) Note, then, that a *feasible program* from $\mathbf{x}$ in D can be redefined as a sequence $\langle x_t, y_{t+1} \rangle$ with $x_0 = \mathbf{x}$, $y_{t+1} = Ax_t$, and (x_t, x_{t+1}) in F for $t \geq 0$. Similarly, an *optimal program* from $\mathbf{x}$ in D can be redefined as a feasible program $\langle x_t^*, y_{t+1}^* \rangle$ from $\mathbf{x}$ such that

$$\liminf_{T \to \infty} \sum_{t=0}^{T} [w(x_t, x_{t+1}) - w(x_t^*, x_{t+1}^*)] \leq 0$$

for every feasible program $\langle x_t, y_{t+1} \rangle$ from $\mathbf{x}$. Keeping this notation in mind, we will, at various points in the next four sections, provide appropriate remarks which will show how forestry theory and intertemporal allocation theory are related.

3. Stationary Forests and a Golden Rule

In this section, we are concerned with the following question: "Suppose the forest on the given piece of land does not change at all from one period to the next, what is the 'best' composition of the forest?" In our terminology, if we look at stationary programs $\langle x, y \rangle$, which stationary program has the maximum per period utility $u(Qc) = u(QB(y-x))$?

We define a *golden rule* to be a triple (x^*, y^*, c^*) with (x^*, y^*) in E, and $c^* = B(y^* - x^*) \geqslant 0$, such that if (x, y) is in E and $c = B(y-x) \geqslant 0$, then

$$u(Qc^*) \geqslant u(Qc).$$

We are interested in characterizing a golden rule.

To this end, assume (A.1), (A.2) and consider the function, h (representing the average product function) as follows:

$$h(a) = [f(a)/a] \qquad \text{for} \quad a > 0; \quad h(0) = 0.$$

Consider now the following problem:

$$\begin{aligned} &\text{Maximize } h(a) \\ &\text{Subject to } a \in [1, 2, ..., N]. \end{aligned} \tag{3.1}$$

Clearly, there is an integer, M, such that (i) $1 < M \leqslant N$, (ii) $h(M) \geqslant h(a)$ for $a \in [1, ..., N]$. Note that $M > 1$, since $h(1) = 0$, while $h(N) > 0$.

Given any solution M to problem (3.1) we can define $\hat{x} = (\hat{x}(0), ..., \hat{x}(N))$ by the following: $\hat{x}(a) = (1/M)$ for $a = 0, 1, ..., M-1$; $\hat{x}(a) = 0$ for $a = M, ..., N$. Defining $\hat{y} = A\hat{x}$, and $\hat{c} = B(\hat{y} - \hat{x})$, we note that $(\hat{x}, \hat{y})$ is in E, and $\hat{c} \geqslant 0$. Furthermore, $Q\hat{c} = h(M)$.

Now, consider any (x, y, c), such that (x, y) is in E, and $c = B(y-x) \geqslant 0$. Then, $c = B(Ax - x) = (x(0) - x(1), ..., x(N-1) - x(N))$, and $Qc = \sum_{a=1}^{N} f(a)\ (x(a-1) - x(a))$. Since $c \geqslant 0$, so $x(a-1) - x(a) \geqslant 0$ for $a = 1, ..., N$; also $f(a) \leqslant ah(M)$ for $a = 1, ..., N$. Thus, $Qc \leqslant h(M) \sum_{a=1}^{N} a(x(a-1) - x(a)) = h(M) \sum_{a=1}^{N} x(a-1) = h(M)$.

We have, therefore, demonstrated that $(\hat{x}, \hat{y}, \hat{c})$ is a golden rule, for any increasing utility function, u. Thus, a golden-rule forest is one in which the total plot of land is divided into M equal sub-plots (M being a solution to problem (3.1)), with one sub-plot each containing input of trees of age a $(a = 0, 1, ..., M-1)$. In each period, trees of age M are cut down, and the sub-plot so cleared is planted with seedlings (age zero trees).

In order to study the properties of optimal forests as we do in the next three sections it becomes crucial to establish a "price-support property" for a golden rule. This means that one associates with the golden rule a shadow price vector such that the utility plus the value of various timber stands carried over, less the value of initial timber stands is maximized at the golden rule among all feasible activities.

The approach taken in the general theory of optimal intertemporal allocation by Gale [6], Brock [1], and McKenzie [11] is to use the Kuhn–Tucker theorem to provide this price-support property. We have chosen to provide a purely constructive proof, which has the advantage that we can identify the shadow prices in terms of the basic data (the production function, f, and the utility function, u) of our model.

Given a solution, M, to problem (3.1), we denote $h(M)$ by β; $u'(\beta)$ by α; $\beta(1,..., N)$ by P; $\beta(0, 1,..., N)$ by q; αq by p.

LEMMA 3.1. *Under* (A.1)–(A.5), *if* (x, z) *is in* F, *then*

$$u[QB(Ax-z)]+pz-px \leqslant u[\beta]. \tag{3.2}$$

Proof. If (x, z) is in F, then $B(Ax-z)=(x(0)-z(1),..., x(N-1)-z(N)) \geqslant 0$. Now, for $a=1,..., N$, we have

$$Q(a)=f(a)=ah(a) \leqslant ah(M)=a\beta. \tag{3.3}$$

So, $Q \leqslant \beta(1,..., N)=P$. Using this information, we have

$$QB(Ax-z) \leqslant PB(Ax-z)=q(Ax-z)=qAx-qz. \tag{3.4}$$

Now, $qA-q=\beta(1, 2,..., N, 0)-\beta(0, 1,..., N)=\beta(1, 1,..., 1, -N)$. Thus, $qAx-qx=\beta(x(0)+\cdots+x(N-1))-\beta Nx(N)=\beta vx=\beta$. Using this information in (3.4), we have

$$QB(Ax-z)-\beta \leqslant qx-qz. \tag{3.5}$$

Multiplying through by α in (3.5), we get

$$\alpha[QB(Ax-z)-\beta] \leqslant px-pz. \tag{3.6}$$

By concavity and differentiability of u, we have

$$u[QB(Ax-z)] \leqslant u(\beta)+\alpha[QB(Ax-z)-\beta]. \tag{3.7}$$

Combining (3.6) and (3.7), we get (3.2). ∎

If there are several solutions to problem (3.1) then there will be several golden rules. We proceed now with the assumption that there is unique solution to problem (3.1):

(A.6) *If* M *and* M' *solve problem* (3.1), *then* $M=M'$.

Under this additional assumption there is a unique golden rule.

THEOREM 3.1. *Under* (A.1)–(A.6), *there is a unique golden rule.*

Proof. Let M be the unique solution to (3.1). Then defining $\hat{x}(a) = (1/M)$ for $a = 0, 1,..., M-1$; $\hat{x}(a) = 0$ for $a = M,..., N$; $\hat{y} = A\hat{x}$, and $\hat{c} = B(\hat{y} - \hat{x})$, we observe that $(\hat{x}, \hat{y}, \hat{c})$ is a golden rule, as demonstrated above.

To prove the uniqueness of the golden rule, suppose (x, y, c) is a golden rule. We will show that $(x, y, c) = (\hat{x}, \hat{y}, \hat{c})$. Note the $c = B(Ax - x) \geqslant 0$, so $(x(a-1) - x(a)) \geqslant 0$ for $a = 1,..., N$. Also, $u(QB(Ax - x)) = u(\beta)$, so $QB(Ax - x) = \beta$. Thus, we have $\sum_{a=1}^{N} f(a)[x(a-1) - x(a)] = \beta$. But, since $f(a) \leqslant ah(a) \leqslant a\beta$, for $a = 1,..., N$, so we have $\sum_{a=1}^{N} f(a)[x(a-1) - x(a)] \leqslant \beta \sum_{a=1}^{N} a[x(a-1) - x(a)] = \beta \sum_{a=1}^{N} x(a-1) = \beta$. Thus, $\sum_{a=1}^{N} f(a)[x(a-1) - x(a)] = \sum_{a=1}^{N} \beta a[x(a-1) - x(a)]$; or $\sum_{a=1}^{N} (\beta a - f(a))[x(a-1) - x(a)] = 0$. Now, since $\beta a \geqslant f(a)$, and $x(a-1) \geqslant x(a)$ for $a = 1,..., N$, so $[\beta a - f(a)][x(a-1) - x(a)] = 0$ for $a = 1,..., N$. We know that $\beta a > f(a)$ for $a = 1,..., M-1, M+1,..., N$; so we must have $x(a-1) = x(a)$ for $a = 1,..., M-1, M+1,..., N$. This means that $c(a) = 0$ for $a = 1,..., M-1, M+1,..., N$. Since $QB(Ax - x) = \beta$ so $c(M) = (1/M)$. This means that $x(a) = x(M) + (1/M)$ for $a = 0, 1,..., M-1$; $x(a) = x(M)$ for $a = M,..., N$. Since $vx = 1$, so $x(M) = 0$, and $x(a) = (1/M)$ for $a = 0, 1,..., M-1$; $x(a) = 0$ for $a = M,..., N$. Thus $x = \hat{x}$; $y = Ax = A\hat{x} = \hat{y}$; $c = B(y - x) = B(\hat{y} - \hat{x}) = \hat{c}$. ∎

The price, p, which "supports" the golden rule in Lemma 3.1 is by no means unique. In fact, we now note an alternative price-support property, using an additional assumption:

(A.7) *$h(a)$ is non-increasing for $M \leqslant a \leqslant N$.*

We note that (A.1), (A.2), and (A.7) made on the production function of trees, f, are consistent with empirical studies (See Clark [2] for details).

This alternative price-support property is used in Theorem 5.2 below to show the optimality of a certain program when the utitlity function is linear. The role of (A.7) in this context is to ensure that trees do not grow "too fast" in some years for $a > M$ (more precisely, to ensure that $f(a+1) - f(a) \leqslant [f(M)/M]$ for $a = M,..., N-1$). Given (A.7), note that for $a = M,..., N-1$ $[f(a+1)/(a+1)] \leqslant [f(a)/a]$, so that $[af(a+1)] \leqslant [(a+1) f(a)] = af(a) + f(a)$. Consequently, $a[f(a+1) - f(a)] \leqslant f(a)$, and $[f(a+1) - f(a)] \leqslant f(a)/a$. Since $[f(a)/a] \leqslant [f(M)/M]$, so $[f(a+1) - f(a)] \leqslant [f(M)/M]$ for $a = M,..., N-1$.

Define $P' = [\beta, 2\beta,..., (M-1)\beta, f(M),..., f(N)]$; $q' = [0, \beta, 2\beta,..., (M-1)\beta, f(M),..., f(N)]$; $p' = \alpha q'$.

COROLLARY 3.1. *Under* (A.1)–(A.7), *if* (x, z) *is in* F, *then*

$$u[QB(Ax - z)] + p'z - p'x \leqslant u[\beta]. \tag{3.8}$$

Proof. If (x, z) is in F, then $B(Ax-z)=(x(0)-z(1),\ldots, x(N-1)-z(N))\geqslant 0$. Now, for $a=1,\ldots, M$, $Q(a)=f(a)=ah(a)\leqslant ah(M)=a\beta$. Also, for $a=M+1,\ldots, N$, $Q(a)=f(a)$. So $Q\leqslant P'$, and

$$QB(Ax-z)\leqslant P'B(Ax-z)=q'(Ax-z)=q'Ax-q'z. \tag{3.9}$$

Now, $q'A-q' = [\beta,\ldots, \beta, f(M+1)-f(M),\ldots, f(N)-f(N-1), -f(N)] \leqslant [\beta,\ldots, \beta, \beta,\ldots, \beta, -f(N)]$, since $f(a+1)-f(a)\leqslant\beta$ for $a=M,\ldots, N-1$, by (A.7). Thus, $q'Ax-q'x\leqslant\beta[x(0)+\cdots+x(N-1)]-f(N)\,x(N)=\beta vx=\beta$. Using this in (3.9),

$$QB(Ax-z)-\beta\leqslant q'x-q'z. \tag{3.10}$$

Multiplying through by α in (3.10), we get

$$\alpha[QB(Ax-z)-\beta]\leqslant p'x-p'z. \tag{3.11}$$

By concavity and differentiability of u, we have

$$u[QB(Ax-z)]\leqslant u(\beta)+\alpha[QB(Ax-z)-\beta]. \tag{3.12}$$

Combining (3.11) and (3.12), we get (3.8). ∎

4. The Existence of an Optimal Program

In this section, we indicate how the existence of an optimal program can be established. This is done for the sake of completeness of the exposition of forestry theory in the undiscounted case; the methods are familiar from the general theory of optimal intertemporal allocation as developed by Gale [6], and modified by McKenzie [11] and Brock [1]. Our exposition will follow the method used by Brock, and we will spell out the details of the steps only when the differences of the forestry model from his warrants it.

Define $v: F\to R$ by $v(x, z)=w(x, z)-w(\hat{x}, \hat{x})$ for (x, z) in F. Also, for (x, z) in F, denote $\delta(x, z)=px-pz-v(x, z)$. By Lemma 3.1, $\delta(x, z)\geqslant 0$ for (x, z) in F. For a feasible program $\langle x_t, y_{t+1}\rangle$ from $\mathbf{x}$ in D, we denote $\delta(x_t, x_{t+1})$ by δ_t, $[(x_1+\cdots+x_t)/t]$ by $\bar{x}_t$, $[(y_1+\cdots+y_t)/t]$ by $\bar{y}_t$. A feasible program $\langle x_t, y_{t+1}\rangle$ is called *good* if there is a real number θ, such that for all $T\geqslant 1$,

$$\sum_{t=1}^{T} v(x_t, x_{t+1})\geqslant\theta. \tag{4.1}$$

To apply the techniques of Brock, the main result to establish is that there is a good program from $\mathbf{x}$ in D.

LEMMA 4.1. *Under* (A.1)–(A.5), *there is a good program from* **x** *in* D.

Proof. Consider the sequence $\langle x_t \rangle$ defined by: $x_0 = \mathbf{x}$, $x_1 = d$,

$$x_t = (\underbrace{(1/M), \ldots, (1/M)}_{(t-1)\text{ times}}, (M-t+1)/M, 0, \ldots, 0) \qquad \text{for} \quad t = 2, \ldots, M;$$

$x_t = \hat{x}$ for $t > M$. Define $y_{t+1} = Ax_t$ for $t \geq 0$. Then, it is easy to check that $\langle x_t, y_{t+1} \rangle$ is a feasible program from **x**. Also since $x_t = \hat{x}$ for all but a finite number of periods, so $\langle x_t, y_{t+1} \rangle$ is a good program from **x**. ▮

Using Lemma 3.1, and following the method of Gale [6], one can show

LEMMA 4.2. *Under* (A.1)–(A.5), *if a feasible program* $\langle x_t, y_{t+1} \rangle$ *is not good, then* $\sum_{t=1}^{T} v(x_t, x_{t+1}) \to -\infty$ *as* $T \to \infty$.

Using the fact that u is concave, and that the golden rule is unique (Theorem 3.1), one can show that good programs satisfy the "average turnpike property."

LEMMA 4.3. *Under* (A.1)–(A.6), *if* $\langle x_t, y_{t+1} \rangle$ *is a good program from* **x** *in* D, *then* $(\bar{x}_t, \bar{y}_t) \to (\hat{x}, \hat{y})$ *as* $t \to \infty$.

Proof. Let $(\bar{x}, \bar{y})$ be any limit point of the sequence $\langle \bar{x}_t, \bar{y}_t \rangle$. Then, clearly, $\bar{x}$ is in D. Also, since $y_{t+1} = Ax_t$, we have $\bar{y}_{t+1} = A(x_0 + \cdots + x_t)/(t+1) = A(x_1 + \cdots + x_{t+1})/(t+1) - A(x_0 - x_{t+1})/(t+1) = A\bar{x}_{t+1} - [A(x_0 - x_{t+1})/(t+1)]$. So, $\bar{y} = A\bar{x}$, and $(\bar{x}, \bar{y})$ is in E. Finally, note that since $B(y_t - x_t) \geq 0$, $B(\bar{y}_t - \bar{x}_t) \geq 0$, and $B(\bar{y} - \bar{x}) \geq 0$; we denote $\bar{c} = B(\bar{y} - \bar{x})$.

Since $\langle x_t, y_{t+1} \rangle$ is good and u is concave, there is a real number θ such that for $t \geq 1$ (denoting $u(k) - u(\beta)$ by $U(k)$ for k in R_+),

$$U(Q\bar{c}_t) \geq (1/t) \sum_{s=1}^{t} U(Qc_s) \geq \theta/t. \tag{4.2}$$

Hence, $U(Q\bar{c}) \geq 0$. But by Lemma 3.1, $U(Q\bar{c}) \leq 0$, so we have $U(Q\bar{c}) = 0$. But then $(\bar{x}, \bar{y}, \bar{c})$ is a golden rule, and by Theorem 3.1, $(\bar{x}, \bar{y}, \bar{c}) = (\hat{x}, \hat{y}, \hat{c})$. Since $(\bar{x}, \bar{y})$ is an arbitrary limit point of the sequence $\langle \bar{x}_t, \bar{y}_t \rangle$, so $(\bar{x}_t, \bar{y}_t) \to (\hat{x}, \hat{y})$ as $t \to \infty$. ▮

Define $\bar{\theta} = \inf(\sum_{t=0}^{\infty} \delta_t : \langle x_t, y_{t+1} \rangle$ is a feasible program from **x** in D). By Lemma 4.1, we know that $\bar{\theta} < \infty$. Using the method of Lemma 5 in Brock [1], one obtains the following result:

LEMMA 4.4. *Under* (A.1)–(A.5), *there is a good program* $\langle x'_t, y'_{t+1} \rangle$ *from* **x** *in* D, *such that* $\sum_{t=0}^{\infty} \delta'_t = \bar{\theta}$.

Using Lemma 4.3 and the method of Theorem 1 of Brock [1], one then obtains the existence of an optimal program.

THEOREM 4.1. *Under* (A.1)–(A.6), *the feasible program* $\langle x_t', y_{t+1}' \rangle$ *from* $\mathbf{x}$ *in* D, *given by Lemma* 4.4, *is an optimal program from* $\mathbf{x}$.

A consequence of Theorem 4.1 and Lemma 4.4 is that the golden-rule program is an optimal program from $\hat{x}$.

COROLLARY 4.1. *Under* (A.1)–(A.6), *the feasible program* $\langle x_t, y_{t+1} \rangle$ *from* $\hat{x}$ *given by* $x_t = \hat{x}$, $y_{t+1} = \hat{y}$ *for* $t \geq 0$ *is an optimal program from* $\hat{x}$.

5. LINEAR UTILITY FUNCTION AND THE FAUSTMANN SOLUTION

In this section, and the next, we will be concerned with the asymptotic properties of optimal programs. For this section, we will assume

(A.8) *u is linear; that is,* $u(k) = m\,k$ *for* k *in* R_+, *where* $m > 0$.

First, we will consider the case in which the land available for forestry is initially empty. In this case, without loss of generality, we can assume that $\mathbf{x} = d$. We will show that, in this case, it is optimal to implement the following "periodic" policy. Let all trees grow up to age M, cut all of them down, and replant the entire forest with seedlings (age zero trees); repeat this process indefinitely. This, of course, is the solution concept proposed by Faustmann [4].

Next, we consider the case in which the land has, initially, a standing forest. In this case, the following obvious modification of the above policy is optimal: initially, cut all trees of age M or higher; thereafter, cut a tree if and only if it is of age M.

Consider the sequence $\langle \tilde{x}_t, \tilde{y}_{t+1} \rangle$ given by

$$\tilde{x}_0 = d, \qquad \tilde{x}_t = A^t d \quad \text{for} \quad t = 1, \dots, M-1; \qquad \tilde{x}_t = \tilde{x}_{t-M} \quad \text{for} \quad t \geq M.$$

$$\tilde{y}_{t+1} = A\tilde{x}_t \qquad \text{for} \quad t \geq 0. \tag{5.1}$$

It can be easily checked that $\langle \tilde{x}_t, \tilde{y}_{t+1} \rangle$ is a feasible program from $\mathbf{x} = d$.

THEOREM 5.1. *Under* (A.1), (A.2), (A.8), *the feasible program* $\langle \tilde{x}_t, \tilde{y}_{t+1} \rangle$ *defined by* (5.1) *is an optimal program from* $\mathbf{x} = d$.

Proof. For any feasible program $\langle x_t, y_{t+1} \rangle$ from $\mathbf{x} = d$, we have (x_{t-1}, x_t) in F for $t \geq 1$. So by using Lemma 3.1, we have

$$Qc_t + qx_t - qx_{t-1} \leq \beta \qquad \text{for} \quad t \geq 1. \tag{5.2}$$

Using (5.2), we have for $T \geqslant 1$,

$$\sum_{t=1}^{T} (Qc_t - \beta) \leqslant \sum_{t=1}^{T} (qx_{t-1} - qx_t) = qx_0 - qx_T \leqslant 0, \tag{5.3}$$

since $qx_0 = qd = 0$. For $T = sM$, where $s = 1, 2, \ldots$

$$\sum_{t=1}^{T} (Q\tilde{c}_t - \beta) = \left(\sum_{t=1}^{T} Q\tilde{c}_t \right) - (sM\beta) = sf(M) - sM\beta = 0. \tag{5.4}$$

Thus, using (5.3) and (5.4) we have for $T = sM$,

$$\sum_{t=1}^{T} (Qc_t - Q\tilde{c}_t) \leqslant 0. \tag{5.5}$$

But (5.5) clearly implies that

$$\liminf_{T \to \infty} \sum_{t=1}^{T} (Qc_t - Q\tilde{c}_t) \leqslant 0, \tag{5.6}$$

which proves that $\langle \tilde{x}_t, \tilde{y}_{t+1} \rangle$ is an optimal program from $\mathbf{x} = d$. ∎

We now consider the case in which $\mathbf{x}$ is an arbitrary vector in D. Let g be the Mth unit-vector in R^N. Define an $(N+1) \times (N+1)$ matrix C as

$$C = \begin{bmatrix} g & 1 \\ I_N & 0 \end{bmatrix}.$$

Consider the sequence $\langle x'_t, y'_{t+1} \rangle$ given by

$$\begin{aligned} &x'_0 = \mathbf{x}, \qquad x'_1(a) = \mathbf{x}(a-1) \qquad \text{for} \quad a = 1, \ldots, M-1 \\ &x'_1(a) = 0 \qquad \text{for} \quad a \geqslant M, \qquad x'_1(0) = \sum_{a=M}^{N} \mathbf{x}(a-1). \\ &x'_t = [CA]^t x'_1 \qquad \text{for} \quad t = 2, \ldots, M+1 \\ &x'_t = x'_{t-M} \qquad \text{for} \quad t \geqslant M+2; \qquad y'_{t+1} = Ax'_t \qquad \text{for} \quad t \geqslant 0 \end{aligned} \tag{5.7}$$

It can be checked that $\langle x'_t, y'_{t+1} \rangle$ is a feasible program from $\mathbf{x}$. We will show that $\langle x'_t, y'_{t+1} \rangle$ is an optimal program from $\mathbf{x}$.

THEOREM 5.2. *Under* (A.1), (A.2), (A.6)–(A.8), *the feasible program* $\langle x'_t, y'_{t+1} \rangle$ *defined by* (5.7) *is an optimal program from* $\mathbf{x} \in D$.

Proof. Note, first, that $\langle x'_t, y'_{t+1}\rangle$ is a good program. To see this, note that for $t \geqslant 1$,

$$u[Qc'_{t+1}] + p'x'_{t+1} - p'x'_t = u[\beta]. \tag{5.8}$$

Hence, $\sum_{t=1}^{T} \{u[Qc'_{t+1}] - u[\beta]\} = p'x_1 - p'x_{T+1} \geqslant -p'x_{T+1} \geqslant -f(N)\,N$, which shows that $\langle x'_t, y'_{t+1}\rangle$ is good.

Now, suppose $\langle x'_t, y'_{t+1}\rangle$ is not optimal. Then, there is some feasible program $\langle x_t, y_{t+1}\rangle$, a real number $r' > 0$, and an integer $T^* \geqslant 1$, such that for $T \geqslant T^*$,

$$\sum_{t=1}^{T} \{u[Qc_t] - u[Qc'_t]\} \geqslant r' \qquad \text{for} \qquad T \geqslant T^*. \tag{5.9}$$

This means that $\langle x_t, y_{t+1}\rangle$ is itself good. Next, we note that

$$\begin{aligned} &u(Qc'_1) + p'x'_1 - p'\mathbf{x} = m\beta[\mathbf{x}(0) + \cdots + \mathbf{x}(M-2)] \\ \text{and}\quad & \\ &u(Qc_1) + p'x_1 - p'\mathbf{x} \leqslant m\beta[\mathbf{x}(0) + \cdots + \mathbf{x}(M-2)] \end{aligned} \tag{5.10}$$

Also, for $t \geqslant 1$, by Corollary 3.1,

$$u(Qc_{t+1}) + p'x_{t+1} - p'x_t \leqslant u(\beta). \tag{5.11}$$

Using (5.8)–(5.11), we have for $T \geqslant T^*$,

$$r' \leqslant \sum_{t=1}^{T} \{u[Qc_t] - u[Qc'_t]\} \leqslant p'x'_T - p'x_T. \tag{5.12}$$

Hence, there is $T^{**} \geqslant T^*$ such that

$$p'\bar{x}'_T - p'\bar{x}_T \geqslant r'/2 \qquad \text{for} \quad T \geqslant T^{**}. \tag{5.13}$$

By Lemma 4.3, using the fact that $\langle x_t, y_{t+1}\rangle$ and $\langle x'_t, y'_{t+1}\rangle$ are good programs, we know that $\bar{x}_t \to \hat{x}$ and $\bar{x}'_t \to \hat{x}$ as $t \to \infty$. But this contradicts (5.13) for large enough T. Hence $\langle x'_t, y'_{t+1}\rangle$ is an optimal program from $\mathbf{x}$. ∎

Remark. In order to clarify the descriptions of the optimal program (given by (5.1) and (5.7)), we present a simple example. Suppose $M = 3$, $N = 4$. Let $\mathbf{x} = (1, 0, 0, 0, 0)$. In this case, an optimal program is given by: $\tilde{x}_1 = (0, 1, 0, 0, 0)$, $\tilde{x}_2 = (0, 0, 1, 0, 0)$, $\tilde{x}_3 = \mathbf{x}$; $\tilde{x}_t = \tilde{x}_{t-3}$ for $t > 3$. This is what (5.1) describes. On the other hand, suppose $\mathbf{x} = (\frac{1}{3}, \frac{1}{6}, \frac{1}{2}, 0, 0)$. Then, an optimal program is given by $x'_1 = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, 0)$, $x'_2 = (\frac{1}{6}, \frac{1}{2}, \frac{1}{3}, 0, 0)$, $x'_3 = \mathbf{x}$, $x'_t = x'_{t-3}$ for $t > 3$. This is what (5.7) describes.

One can relate the results of this section to the theory of the "von

Neumann facet" as developed by McKenzie [11]. Let $\phi(p)=\{(x,z)$ in F: $w(x,z)+pz-px=w(\hat{x},\hat{x})]$. Equivalently, in terms of the notation introduced in Section 4, $\phi(p)=\{(x,z)$ in F: $v(x,z)+pz-px=0\}$ or $\phi(p)=\{(x,z)$ in F: $\delta(x,z)=0\}$. Following McKenzie, one can call $\delta(x,z)$ the "value loss" suffered by operating at the "technology" pair (x,z) (compared to the golden-rule point $(\hat{x},\hat{x})$) when valuation is at the price, p. (Note that $\delta(x,z)\geqslant 0$ for all (x,z) in F.) We can then call $\phi(p)$ the "facet" of the "technology" set, F, which suffers zero value loss (compared to the golden-rule point $(\hat{x},\hat{x})$), when valuation is at the price, p. (Given another price vector, for example, p', at which $w(x,z)+p'x-p'x\leqslant w(\hat{x},\hat{x})$ for all (x,z) in F, one can analogously define a facet $\phi(p')$, which suffers zero value loss compared to the golden-rule point when valuation is at the price, p'.)

The program described by (5.1) is one which starts from virgin land and follows a policy in which a tree is cut if and only if it is of age M. Consequently, it suffers zero value-loss for every time period. Any other program has non-negative value-losses in each period. This, together with the fact that the program described by (5.1) periodically returns to a "virgin-land" state, and hence a zero state valuation, while any other program has non-negative state valuation at each date, means that the given program is optimal according to our criterion.

The idea of the proof of Theorem 5.2 is similar. One tries to construct a program which suffers zero value loss in all periods. However, since $\mathbf{x}$ is now an arbitrary point in D, there might be no z with the property that $(\mathbf{x},z)$ is in $\phi(p)$; that is, one might inherit a "badly-managed" forest. The best one can hope for, then, is to construct a program which suffers zero value-loss in all periods *after the initial period.* A natural "candidate" program is one in which all trees of age greater than or equal to M are cut down initially. Thereafter, a tree is cut if and only if it is of age M. This is described in (5.7). Notice that along the program all trees of age greater than or equal to M should be cut down in the intitial period. How do we know that such an initial action is optimal? The answer is that, in general, we do not know for sure that this is optimal; if trees of age greater than M can grow "quite fast" for some periods, it is not obvious why, *having inherited them*, one sould cut them down immediately. This explains our need for (A.7), which ensures that trees beyond age M do not grow "too fast." Given this, it is possible to construct another price vector, p' (Corollary 3.1) at which the constructed program suffers zero value-loss for all periods $t\geqslant 1$, and at which the program suffers *minimal* (although possibly positive) value loss in the initial period compared to any other program *starting from the same initial forest.*

There is another aspect in which the proof of Theorem 5.2 differs from that of Theorem 5.1. The candidate program need not periodically return to a state with zero input valuation. Thus, one has to be careful about the asymptotic levels of input valuation on the candidate program relative to an alternative comparison program. Here, we use the "average turnpike property" of good programs, which was used earlier to prove the existence of an optimal program in Section 4. Since this requires a unique golden-rule program, and (A.6) ensures that this is the case, our use of (A.6) in Theorem 5.2 (though not in Theorem 5.1) is understandable.

6. Strictly Concave Utility Function and the Asymptotic Turnpike Solution

For this section, we strengthen assumption (A.5) to

(A.9) *u is strictly concave.*

Under this additional assumption, we will show that any optimal program from $\mathbf{x}$ in D must converge to the golden rule. Thus, the golden rule serves as the turnpike, and any optimal program obeys asymptotically the turnpike solution. Sections 5 and 6 show that there is a significant qualitative difference in the asymptotic behavior of optimal programs depending on the concavity assumption that is made on the utilty function.

The ideas leading to this result can be explained simply and related to the existing literature. It is clear from the proof of Lemma 3.1 (particularly step (3.7)) that with a strictly concave utility function, the only processes which avoid value-loss (that is, are on the facet, $\phi(p)$) are those which harvest trees at age M in the quantity $h(M)$. This is the content of Lemma 6.1.

This leads to the so-called "value-loss result," which we state as Lemma 6.2. It says that if the harvest is uniformly at least a certain positive "distance" away from the golden-rule harvest, then the value-loss is uniformly at least a certain positive number. On the other hand a good program, and hence an optimal program, can only suffer a sequence of value-losses (δ_t), such that the sum of such losses is finite. This means that value-losses converge to zero over time, and by the value-loss result, optimal harvests converge to the golden-rule harvest (Lemma 6.3). This part of the story corresponds to McKenzie's [11] analysis of the convergence of optimal paths *to* the von Neumann facet.

The next step is to show that a feasible program lying *on* the facet, $\phi(p)$, will converge in terms of the input levels to the golden rule input level. That is, a feasible program $\langle x_t, y_{t+1} \rangle$ satisfying (by Lemma 6.1)

$$B(Ax_t - x_{t+1}) = \hat{c} \qquad \text{for} \quad t \geqslant 0$$

must have $x_t \to \hat{x}$ as $t \to \infty$. This result can be shown by following the general technique of McKenzie [11], using the theory of matrix pencils. (See Sect. III of his paper dealing with "Convergence on the von Neumann Facet," particularly Theorem 2, and also Lemma 8 in his Section V.) We provide in Lemma 6.4 a direct proof for our special case; this is done to keep the exposition self-contained. The asymptotic convergence result for optimal programs is then summarized in Theorem 6.1.

Recall that the Mth unit vector in R^N is denoted by g, and note that $(g/M) = \hat{c}$.

LEMMA 6.1. *Under* (A.1)–(A.4), (A.6), (A.9), *if* (x, z) *is in* F *and* $\delta(x, z) = 0$, *then*

$$B(Ax - z) = \hat{c}. \tag{6.1}$$

Proof. Using the method of proof of Lemma 3.1, if $\delta(x, z) = 0$, then (i) $QB(Ax - z) = qAx - qz$. Using (A.9), we note that if $\delta(x, z) = 0$, then (ii) $QB(Ax - z) = \beta$. Using the method of proof of Theorem 3.1, (i) can be satisfied only if

$$x(a-1) = z(a) \qquad \text{for} \quad a = 1,..., M-1, M+1,..., N. \tag{6.2}$$

Given (6.2), (ii) can be satisfied only if

$$x(M-1) - x(M) = (1/M). \tag{6.3}$$

Combining (6.2) and (6.3), we get (6.1). ∎

For (K, K') in $R^L \times R^L$, we define a distance function

$$J(K, K') = \sum_{i=1}^{L} |K_i - K'_i|.$$

We now establish a "value loss" result of the type proved by Radner [14] and McKenzie [10, 11].

LEMMA 6.2. *Under* (A.1)–(A.4), (A.6), (A.9) *given* $\gamma > 0$, *there is* $\delta > 0$, *such that if* (x, z) *is in* F, *and* $J(B(Ax - z), \hat{c}) \geq \gamma$, *then* $\delta(x, z) \geq \delta$.

Proof. Suppose, on the contrary, there is a sequence (x^s, z^s) in F, such that $J(B(Ax^s - z^s), \hat{c}) \geq \gamma$, but $\delta(x^s, z^s) \to 0$ as $s \to \infty$. Since (x^s, z^s) is in a compact set, there is a subsequence $(x^{s'}, z^{s'})$ converging to (x^*, z^*), where (x^*, z^*) is in F. Since $J(B(Ax^{s'} - z^{s'}), \hat{c}) \geq \gamma$, $J(B(Ax^* - z^*), \hat{c}) \geq \gamma$. This means that $\delta(x^*, z^*) > 0$, by Lemma 6.1. Since $(x^{s'}, z^{s'})$ converges to (x^*, z^*), $\delta(x^{s'}, z^{s'}) \to \delta(x^*, z^*)$. But since $\delta(x^{s'}, z^{s'}) \to 0$ as $s' \to \infty$, $\delta(x^*, z^*) = 0$, a contradiction. This proves the lemma. ∎

LEMMA 6.3. *Under* (A.1)–(A.4), (A.6), (A.9), *if* $\langle x_t, y_{t+1}\rangle$ *is a good program for* $\mathbf{x}$ *in* D, *then*

$$\sum_{t=0}^{\infty} \delta_t < \infty, \tag{6.4}$$

$$\delta_t \to 0 \quad \text{as} \quad t \to \infty, \tag{6.5}$$

$$J(c_t, \hat{c}) \to 0 \quad \text{as} \quad t \to \infty. \tag{6.6}$$

Proof. We have for $T \geqslant 1$,

$$\sum_{t=1}^{T} v(Qc_t) = \sum_{t=1}^{T} (px_{t-1} - px_t) - \sum_{t=1}^{T} \delta_{t-1} = p\mathbf{x} - px_T - \sum_{t=1}^{T} \delta_{t-1}.$$

Since $\langle x_t, y_{t+1}\rangle$ is a good program, there is a real number θ such that for $T \geqslant 1$, we have

$$\sum_{t=1}^{T} \delta_{t-1} \leqslant p\mathbf{x} - px_T - \theta \leqslant p\mathbf{x} + |\theta|.$$

Since $\delta_t \geqslant 0$ for $t \geqslant 0$ by Lemma 3.1, so (6.4) follows. Using (6.4), one obtains (6.5) immediately. Using (6.5), and Lemma 6.2, we obtain (6.6). ∎

LEMMA 6.4. *Under* (A.1)–(A.4), (A.9), *if* $\langle x_t, y_{t+1}\rangle$ *is a good program from* $\mathbf{x}$ *in* D, *then*

$$J(x_t, \hat{x}) \to 0 \quad \text{as} \quad t \to \infty. \tag{6.7}$$

Proof. Using Lemma 6.3 and given $\gamma > 0$, we can choose $T^* < \infty$ such that for $t \geqslant T^*$, $J(c_t, \hat{c}) < (\gamma/N^4)$. We will show that

$$J(x_{s+M}, \hat{x}) \leqslant \gamma \qquad \text{for each } s > T^*. \tag{6.8}$$

Note that since $J(c_t, \hat{c}) < (\gamma/N^4)$, for $t > T^*$, $(x_t(a-1) - x_{t+1}(a)) < (\gamma/N^4)$ for $a = 1,..., M-1$, $M+1,..., N$. Also, $(x_t(M-1) - x_{t+1}(M)) > (1/M) - (\gamma/N^4)$ for $t > T^*$. Using these two pieces of information, we have for $t > T^*$,

$$x_{t+1}(0) = \sum_{a=1}^{N} (x_t(a-1) - x_{t+1}(a)) > (1/M) - (\gamma/N^3). \tag{6.9}$$

Now, for $r = 1,..., M-1$ and $t > T^*$,

$$\begin{aligned} x_{t+r+1}(r) &= x_{t+1}(0) + \sum_{a=1}^{r} (x_{t+a+1}(a) - x_{t+a}(a-1)) \\ &> (1/M) - (\gamma/N^3) - r(\gamma/N^4) \\ &> (1/M) - (2\gamma/N^3) \geqslant (1/M) - (\gamma/N^2). \end{aligned}$$

Using this information, together with (6.9), we have for $t > T^*$,

$$x_{t+r+1}(r) > (1/M) - (\gamma/N^2) \qquad \text{for} \quad r = 0, 1, ..., M-1. \tag{6.10}$$

Now pick any $s > T^*$. Then since $vx_{s+M} = 1$, using (6.10),

$$\sum_{a=M}^{N} x_{s+M}(a) < 1 - M[(1/M) - (\gamma/N^2)] = M\gamma/N^2. \tag{6.11}$$

Also, using (6.10) we have

$$x_{s+M}(a) < [(1/M) + (\gamma/N)] \qquad \text{for} \quad a = 0, 1, ..., M-1. \tag{6.12}$$

For if $x_{s+M}(a) \geqslant (1/M) + (\gamma/N)$ for some a satisfying $1 \leqslant a \leqslant M-1$, then

$$\begin{aligned} \sum_{a=0}^{N} x_{s+M}(a) \geqslant \sum_{a=0}^{M-1} x_{s+M}(a) &> [(1/M) + (\gamma/N)] \\ &\quad + (M-1)[(1/M) - (\gamma/N^2)] \\ &= 1 + (\gamma/N) - (M-1)(\gamma/N^2) \geqslant 1, \end{aligned}$$

which is a contradiction. This establishes (6.12). Now, using (6.10) and (6.12) we can conclude that

$$|x_{s+M}(a) - (1/M)| < (\gamma/N) \qquad \text{for} \quad a = 0, 1, ..., M-1. \tag{6.13}$$

Using (6.11) and (6.13) we finally have, for $M \leqslant N-1$,

$$\begin{aligned} J(x_{s+M}, \hat{x}) &= \sum_{a=0}^{M-1} |x_{s+M}(a) - (1/M)| + \sum_{a=M}^{N} |x_{s+M}(a)| \\ &< (M\gamma/N) + (M\gamma/N^2) \leqslant [1 - (1/N^2)]\,\gamma < \gamma. \end{aligned}$$

For $M = N$, $J(x_{s+M}, \hat{x}) < \gamma$, using (6.13). This confirms (6.8) and proves (6.7). ▌

THEOREM 6.1. Under (A.1)-(A.4), (A.6), (A.9), *if* $\langle x_t, y_{t+1} \rangle$ *is an optimal program from* $\mathbf{x}$ *in* D, *then* x_t *converges to* $\hat{x}$ *as* t *tends to infinity.*

Proof. By Lemmas 4.1 and 4.2, an optimal program must be good. Hence, by Lemma 6.4, the result follows. ▌

7. CONCLUDING REMARKS

In this section, we comment briefly on how our analysis of the forest management problem can be generalized in several directions.

First, note that we have implicitly assumed the costs of planting and harvesting to be zero. These costs may be treated in several ways. One would be to consider wage, w, to be fixed in terms of timber, and to assume immediate replanting after a tree is cut down (for example, for soil conservation or for seizing the opportunity to utilize the land released immediately). Then one may simply redefine the function by replacing $f(a)$ by $[f(a) - w]$. Our results would then still carry over.

Second, the analysis of the linear utility function (Sect. 5) may be applied to the case of a competitive manager of a forest, where m is the constant profit per unit of timber, and the interest rate is zero. Similarly, the analysis of the strictly concave utility function (Sect. 6) may be applied to the case of a monopolistic manager of a forest where u is the profit function, and the interest rate is zero. Of course, the profit function may not be increasing throughout, as (A.3) demands. However, so long as the profit function is increasing on $[0, N]$, our results will carry over.

Third, we have ignored in our analysis the problem of "forest-thinning" and how this could affect the function $f(a)$. To the extent that the Faustmann solution starting from an empty tract of land involves a forest full of mature trees (of the same age M) just before they are cut down, while the golden-rule solution does not, the problem of forest-thinning will be more serious for programs following periodic Faustmann solutions than for those following asymptotically golden-rule solutions. Thus, one may conjecture that, when the aspect of "forest-thinning" is properly accounted for in our analysis, the Faustmann solution need not be optimal, even with a linear utility function.

Finally, an important question not discussed in this paper is the following: Would the two types of asymptotic results (depending on the linearity or the strict concavity of the utility function) continue to hold if future utilities are discounted? Kemp and Moore [9] conjecture that this will indeed be the case, and carry out numerical analysis of special cases to support their conjecture in a continuous-time analysis of the forest-management problem. On the other hand, the recent literature on optimal intertemporal allocation when future utilities are discounted suggests that optimal programs may follow an asymptotic turnpike solution only when the discount rate, ρ, is "sufficiently small." (These results are proved both in discrete-time and continuous-time frameworks.) Our own investigation of this question, in a discrete-time framework, shows that there is a basic difference in the analysis of the discounted case, compared with the undiscounted case, when the utility function is strictly concave. Discussion of this difference is, of course, beyond the scope of the present paper. The interested reader is referred to Mitra and Wan [12] and the references cited there for details.

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